20071017 Princeton Geometry, Representation theory, and moduli Seminar

Perverse coherent sheaves on a blowup surface II

X : nonsingular projective surface $/\mathbb{C}$

$x \in X$

$$\begin{array}{ccc} p : \hat{X} & \longrightarrow & X \\ \cup & & \Psi \\ C & \longrightarrow & p \end{array}$$

blowup at p

Bridgeland considered more general setting

- birational
- $Rp_*(\mathcal{O}_{\hat{X}}) = \mathcal{O}_X$
- relative dim. one

$\mathcal{A} = D(\hat{X})$: derived category of coherent sheaves on $\hat{X}$

$\mathcal{B} = D(X) \xrightarrow{Lp^*} D(\hat{X})$ full subcategory

$\mathcal{C} = \mathcal{B}^{\perp} = \{ a \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(b, a) = 0 \quad \forall b \in \mathcal{B} \}$

$= \{ a \in \mathcal{A} \mid Rp_*(a) = 0 \}$

$= \{$ direct sums of $\mathcal{O}_C(-1)[m] \}$ (Orlov)

$\mathcal{A} = \langle \mathcal{C}, \mathcal{B} \rangle$

$\Rightarrow \mathrm{Per}(\hat{X}/X) \subset D(\hat{X})$ perverse coherent sheaf

obtained by "gluing" cores of $\mathcal{B}$ & $\mathcal{C}$ in a different way

Def. (Bridgeland)

$E \in D(\hat{X})$ is perverse coherent $(\in \mathrm{Per}(\hat{X}/X))$

(i) $H^i(E) = 0$ for $i \neq -1, 0$

(ii) $p_*(H^{-1}(E)) = 0$, $\quad R^1 p_*(H^0(E)) = 0$

(iii) $\mathrm{Hom}(H^0(E), c) = 0 \quad \forall c \in \mathcal{C} \cap \mathrm{Coh}\,\hat{X}$

- $Rp_*E \in \mathrm{Coh}\,X$ Thus $\mathrm{Perv}(\hat{X}/X)$ is close to $\mathrm{Coh}\,X$.
- $\mathrm{Perv}(\hat{X}/X)$ is an abelian category.

Example

$y \in C \qquad 0 \to \underset{\mathrm{Perv}(\hat{X}/X)}{\mathcal{O}(-C)} \to \mathcal{I}_y \to \underset{\mathrm{Perv}(\hat{X}/X)[1]}{\mathcal{O}_C(-1)} \to 0$

But $\mathcal{I}_y \notin \mathrm{Perv}(\hat{X}/X)$

exchange L, R $\qquad 0 \to \mathcal{O}_C(-1) \to E \to \mathcal{O}(-C) \to 0 \qquad (\star)$

or $\qquad 0 \to \underset{\mathrm{Perv}(\hat{X}/X)[1]}{E} \to \mathcal{O}(-C) \to \mathcal{O}_C(-1)[1] \to 0$

Since $\dim \mathrm{Ext}^1(\mathcal{O}(-C), \mathcal{O}_C(-1)) = 1$, $\star$ is unique up to isom.

$\therefore$ Moduli of Perv. coh. "ideal" sheaves on $\hat{X}$ $\cong X$

Rem. Bridgeland considered 3-dim'l situation

- Moduli of Perv. coh "ideal" sheaves $\cong X^+$ flop
- $D(X) \cong D(X^+)$ given by FM transform w.r.t. the universal family

Thus it is natural to expect
moduli spaces of perv. coherent sheaves on $\hat{X}$
should be close to moduli spaces
of coh. sheaves on X
More precisely, put the "stability" condition
so that the above is true.

H : ample line b'dle on X

Assume H-stable $\Leftrightarrow$ H-semistable

<u>Def.</u> $E \in \mathrm{Per}(\hat{X}/X)$ is stable

$\overset{\text{def}}{\Longleftrightarrow}$ 0) $E \in \mathrm{Coh}\,\hat{X}$ (i.e. $H^{-1}(E) = 0$)

1) $p_* E$ is an H-stable torsion-free sheaf.

<u>Rem.</u> This, in fact, comes from more conceptual definition via comparison of "Hilbert" polynomials on subobjects.

little bit more tractable definition

stable

$\Leftrightarrow$ 0) $E \in \mathrm{Coh}\,\hat{X}$

1) $\mathrm{Hom}(\mathcal{O}_C, E) = 0$ ($\Leftrightarrow$ torsion-freeness of $p_* E$)

2) $\mathrm{Hom}(E, \mathcal{O}_C(-1)) = 0$ perverse cond. (iii)

3) $p_* E$ is H-stable

Under the wall-crossing 1), 2) are violated.

$M^p_H(c)$ = moduli space of stable perverse coherent sheaves on $\hat{X}$

$M_m(c) := \{ E \in \mathrm{Coh}\,\hat{X} \mid E(-mC) \in M^p_H(c) \}$

$$(c_1, C) = 0 \qquad \underset{\substack{\shortparallel \\ \text{moduli on } X}}{M_0(c)} \qquad M_1(c) \qquad \cdots\cdots \qquad \underset{\substack{\shortparallel \\ \text{moduli on } \hat{X}}}{M_N(c)} \qquad N \gg 0$$

o 1^{st} assertion is not so difficult:

<u>Lemma</u> $M_0(c) \equiv M^p_H(c) \cong M_{X,H}(c)$ if $(c_1, C) = 0$

$$E \longmapsto p_* E$$

(sketch)

Consider $p^* p_* E \xrightarrow{\phi} E$.

Perverse $\Rightarrow$ surjective

A little bit of thought $\Rightarrow$ $\mathrm{Ker}\,\phi \cong \mathcal{O}_C(-1)^{\oplus p}$ $(p \geq 0)$

Chern class cond. $\Rightarrow$ $p = 0$ //

— Two constructions of moduli spaces:

1) Following Bridgeland, take the quotient of Quot w.r.t. $p^* \mathcal{O}_X(1)$.

2) Consider framed sheaves on $\mathbb{P}^2$, $\hat{\mathbb{P}}^2$, and use monad description (cf King)

Adv. of 1)

— Works for any surface

Disadv. of 1)

— The usual moduli & perverse moduli cannot be constructed simultaneously.

Only indivisual $M_k(c)$.

2) (E, Φ) : framed $\overset{\text{torsion free}}{\text{sheaf}}$ on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$

$$\Phi : E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$$

$$\text{framed moduli.} \cong \left\{ \begin{tikzcd} V \arrow[loop left, "B_1"] \arrow[loop right, "B_2"] \arrow[d, shift left, "j"] \\ W \arrow[u, shift left, "i"] \end{tikzcd} \quad \begin{array}{l} a)\ [B_1, B_2] + ij = 0 \\ b)\ S \subset V \text{ s.t.} \\ \quad \mathrm{Im}\, i \subset S,\ B_\alpha(S) \subset S \end{array} \right\} \Big/ GL(V)$$

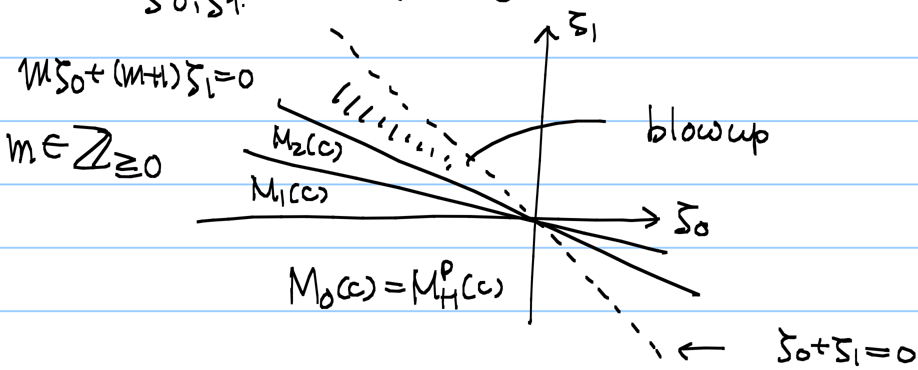
where V, W : vector spaces of $\dim = c_2(E)$, $\operatorname{rank} E$

○ variant on $\hat{\mathbb{P}}^2$

$$\begin{tikzcd} V_0 \arrow[rr, shift left, "d"] & & V_1 \arrow[ll, shift left=2, "B_1"'] \arrow[ll, shift left=1, "B_2"] \arrow[dl, "j"] \\ & W \arrow[ul, "i"] & \end{tikzcd} \qquad \begin{array}{l} B_1 d B_2 - B_2 d B_1 + ij = 0 \\ \left(\begin{array}{l} \dim V_0 = \dim V_1 = c_2 \\ \dim W = \operatorname{rank} \end{array} \right. \end{array}$$

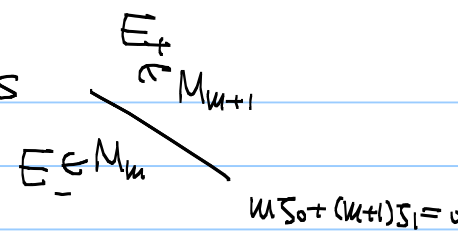
Taking GIT quot. w.r.t. the trivial line b'dle with nontrivial action: $GL(V_0) \times GL(V_1) \xrightarrow{\chi} \mathbb{C}^*$

$$\chi = \chi_{\zeta_0, \zeta_1} = (\det g_0)^{\zeta_0} \cdot (\det g_1)^{\zeta_1}$$



We can construct the moduli for param. on the wall.

○ Wall-crossing and coherent systems



Suppose $E_- \in M_m, \notin M_{m+1}$

$E_- \in M_m \Rightarrow \quad \mathrm{Hom}(E_-, \mathcal{O}_C(-m-1)) = 0$ by definition

$\quad (\mathrm{Hom}(\mathcal{O}_C(-m), E_-) = 0)$

$E_+ \in M_{m+1} \Rightarrow \quad \mathrm{Hom}(\mathcal{O}_C(-m-1), E_+) = 0$ ← This should be violated!

$\quad (\mathrm{Hom}(E_+, \mathcal{O}_C(-m-2)) = 0)$

Then

$\exists \quad 0 \to \mathcal{O}_C(-m-1)^{\oplus p} \to E_- \to E' \to 0$ with $E' \in M_m \cap M_{m+1}(C')$

(JH filtration w.r.t. the stab. param. on the wall)

Conversely if $E_+ \in M_{m+1}, \notin M_m$

Then

$\exists \quad 0 \to E' \to E_+ \to \mathcal{O}_C(-m-1)^{\oplus p} \to 0$ with $E' \in M_m \cap M_{m+1}(C')$

This is a typical picture in the wall-crossing

L term $\longleftrightarrow$ R term in a short exact seq.

○ Brill-Noether locus

$$M_m^p = \{ E_- \in M_- \mid \dim \mathrm{Hom}(\mathcal{O}_C(-m-1), E_-) = p \}$$

$$M_{m+1}^p = \{ E_+ \in M_+ \mid \dim \mathrm{Hom}(E_+, \mathcal{O}_C(-m-1)) = p \}$$

Thus under the wall-crossing :

$$\begin{array}{ccc} M_m & \dashleftarrow\dashrightarrow & M_{m+1} \\ \cup & & \cup \\ M_m \setminus \bigcup_{p>0} M_m^p & \cong & M_{m+1} \setminus \bigcup_{p>0} M_{m+1}^p \end{array}$$

Further M_m^p is a Grassmann b'dle of p-planes in $\mathrm{Ext}^1(E', \mathcal{O}_C(-m-1))$ over $M_m \cap M_{m+1}(c') = M_m^0(c')$

Similarly $M_{m+1}^p = Gr(p, \mathrm{Ext}^1(\mathcal{O}_C(-m-1), E'))$.

$$\downarrow$$

$$M_m \cap M_{m+1}(c') = M_m^0(c')$$

By dimension calculation :

$$\overline{M_m^p} = \bigcup_{q \geq p} M_m^q \quad : \text{ singular !}$$

Rem. Ass. is not true for (-2)-curve case $\Longrightarrow$ One of Key ingredients of the geometric construction of Kashiwara crystal

Note $\operatorname{Ext}^1(E', \mathcal{O}_C(-m-1))$ extends to $M_m(c')$
(but not to $M_{m+1}(c')$)

$$
\begin{array}{ccc}
Gr(p, \operatorname{Ext}^1(E', \mathcal{O}_C(-m-1))) & \longrightarrow & M_m^{\geq p}(c) \\
\downarrow & & \\
M_m(c') & &
\end{array}
$$

resolution of singularities

Using this diagram, the wall-crossing formula of the integration can be done recursively.
It is algorithmic, but the final expression is difficult to handle so far .....